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SHORTER COMMUNICATIONS

THE TEMPERATURE DISTRIBUTION IN AN INFINITE MEDIUM RESULTING FROM A PLANE SOURCE OF FINITE DURATION

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NOMENCLATURE

c, heat capacity;
C_n, constant;
$g(y)$, temperature distribution function at time t_0;
k, thermal conductivity;
M_n, solution function;
p, period of heat generation;
Q', amount of energy generated instantaneously per unit area;
Q, amount of heat generated per unit area per unit time;
y, position from plane source;
t, time;
T, temperature;
ρ, density;
η, transformation variable.

INTRODUCTION

THE TEMPERATURE distribution in an infinite slab during and following heat generation by a plane source of finite duration can be found by integration of the solution of a unit instantaneous plane source. Carslaw and Jaeger [1] treat this problem for the case of continuous constant generation rate. In this paper, a new expression for the instantaneous plane source is obtained. The temperature distribution following the end of heating is more conveniently obtained by integration of this expression than the exponential expression of Carslaw and Jaeger. Further, the treatment is extended to time varying heat generation functions and an example is presented.

SOLUTION TO PLANE SOURCES AND SINKS IN AN INFINITE MEDIUM

The energy equation for the region around an instantaneous plane source generated at time $t = t'$ can be written as

$$\rho c \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial y^2} \quad t > t' \tag{1}$$

with boundary conditions

$$T(t, \infty) = 0 \tag{2a}$$

$$\frac{\partial T}{\partial y}(t, 0) = 0 \tag{2b}$$

and global conservation of energy expressed as

$$\int_{-\infty}^{\infty} \rho c T \, dy = Q' \tag{2c}$$

where Q' is defined as the strength of the source.

The solution to the above equations can be given in the form of a series

$$T(t, y) = \sum_{n=0}^{\infty} \frac{C_n M_n e^{-\eta^2/2}}{t^{n+\frac{1}{2}}} \tag{3}$$

where the C_n are constant coefficients and

$$\eta = y\left(\frac{\rho c}{2kt}\right)^{\frac{1}{2}} \tag{3a}$$

$$M_n = \sum_{j=0}^{n} \frac{(-1)^j n! \eta^{2j}}{2^j (n-j)! j! (j-\frac{1}{2})!} \tag{3b}$$

where

$$(j - \tfrac{1}{2})! = \tfrac{1}{2} . 1\tfrac{1}{2} . 2\tfrac{1}{2} . 3\tfrac{1}{2} \ldots (j - \tfrac{1}{2})$$

$$(0 - \tfrac{1}{2})! = 1$$

The derivation of the above is completely analogous to a similar derivation given in [2].

The constants, C_n, in equation (3) can be evaluated if it is given that at a certain time, $t = t_0$, the temperature distribution function is $g(y)$, where $g(y)$ is an even polynomial expression of $e^{-by^2/2}$.

To evaluate the constants, we make use of the following properties of M_n

$$\int_0^{\infty} M_n M_m e^{-\eta^2/2} \, d\eta = 0, \qquad n \neq m \tag{4}$$

$$= \frac{n!(2\pi)}{(n-\frac{1}{2})!} \qquad n = m \tag{5}$$

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The proof of the above is analogous to that given in [2].

Thus, we may write

$$g(y) = \sum_{n=0}^{\infty} \frac{C_n M_n e^{-\eta^2/2}}{t_0^{n+\frac{1}{2}}} \quad (6)$$

where η is evaluated at $t = t_0$.

Multiplying the above by M_n and integrating between the limits $-\infty$ and ∞, we get the expression

$$\frac{n!(2\pi)^{\frac{1}{2}} C_n}{(n-\frac{1}{2})! t_0^{n+\frac{1}{2}}} = \int_{-\infty}^{\infty} M_n g(y)\, d\eta. \quad (7)$$

For instance, if $g(y) = \exp[-(\rho c y^2/4kt_0')]$, the solution for C_n is

$$C_n = \frac{(n-\frac{1}{2})! t_0^{n+\frac{1}{2}}}{(2\pi)^{\frac{1}{2}} n!} \int_{-\infty}^{\infty} M_n \exp[-(\rho c y^2/4kt_0')]\, d\eta$$

$$= \frac{(n-\frac{1}{2})! t_0^{n+\frac{1}{2}}}{(2\pi)^{\frac{1}{2}} n!} \int_{-\infty}^{\infty} M_n \exp[-(t_0 \eta^2/2t_0')]\, d\eta$$

$$C_n = \frac{(n-\frac{1}{2})! t_0^{n+\frac{1}{2}}}{n!} \sum_{j=0}^{n} \frac{(-1)^j n!}{(n-j)!\, j!\, (t_0/t_0')^{j+\frac{1}{2}}}$$

and upon simplification of the above, we get

$$C_n = \frac{(n-\frac{1}{2})! t_0'^{\frac{1}{2}}}{n!} (t_0 - t_0')^n \quad (8)$$

the expression for T becomes

$$T = t_0'^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(n-\frac{1}{2})!\,(t_0 - t_0')^n}{n!\, t^{n+\frac{1}{2}}} \times \sum_{j=0}^{n} \frac{(-1)^j n!}{(n-j)!\, j!\, (j-\frac{1}{2})!} \left(\frac{\eta^2}{2}\right)^j e^{-\eta^2/2}. \quad (9)$$

It is to be noted that when $t_0 = t_0'$, the solution reduces to [since $g(y)$ is dimensionless so is T]

$$T = t_0'^{\frac{1}{2}} \frac{e^{-\eta^2/2}}{t^{\frac{1}{2}}}.$$

This is the solution of a simple, instantaneous plane source of strength $(4\pi k \rho c t_0')^{\frac{1}{2}}$ which was generated at time $t = t_0'$. This means that equation (9) is the solution of a source which is generated at time $t' = t_0 - t_0'$. Taking t' as the time elapsed before the instantaneous source generates, we get the following expression for a simple unit source.

$$T(t, y) = \frac{1}{(4\pi k \rho c)^{\frac{1}{2}}} \sum_{n=0}^{\infty} \frac{(n-\frac{1}{2})!}{n!} \frac{t'^n}{t^{n+\frac{1}{2}}} e^{-\eta^2/2} M_n. \quad (10)$$

The above solution is equivalent to the solution of [1] which can be written as

$$T(t, y) = \frac{1}{(4\pi k \rho c)^{\frac{1}{2}}} \frac{\exp\{-[\rho c y^2/4k(t - t')]\}}{(t - t')^{\frac{1}{2}}}. \quad (11)$$

The proof of equivalence is analogous to that given in [2].

The advantage of equation (10) over equation (11) lies in the integrability of the former with respect to t'. The importance of this property can be seen in the following section.

APPLICATION OF SERIES SOLUTION OF AN INSTANTANEOUS HEAT SOURCE

If the heat source is of finite duration, p, and strength $Q(t')$ per unit area, the solution to the temperature distribution is the summation of the effects of the heat source at differential increments Δt, each of which may be considered as an instantaneous plane source generated at time t'. The expression for T is

$$T(t, r) = \frac{1}{(4\pi k \rho c)^{\frac{1}{2}}} \int_0^t Q(t') \times \sum_{n=0}^{\infty} \frac{(n-\frac{1}{2})!\, t'^n}{n!\, t^{n+\frac{1}{2}}} e^{-\eta^2/2} M_n\, dt'. \quad (12)$$

The above is easily integrated when Q is a polynomial expression of t'^i where i is arbitrary. Thus, when $Q = Bt'^i$, we get upon integration of the above

$$T(t, r) = \frac{B t^{i+\frac{1}{2}}}{(4\pi k \rho c)^{\frac{1}{2}}} \sum_{n=0}^{\infty} \frac{(n-\frac{1}{2})!}{n!(n+1+i)} M_n e^{-\eta^2/2}, \quad t \leqslant p, \quad i > -1 \quad (13a)$$

$$T(t, r) = \frac{B p^{i+\frac{1}{2}}}{(4\pi k \rho c)^{\frac{1}{2}}} \times \sum_{n=0}^{\infty} \frac{(n-\frac{1}{2})!}{n!(n+1+i)} \left(\frac{p}{t}\right)^{n+\frac{1}{2}} M_n e^{-\eta^2/2}, \quad t > p, \quad i > -1 \quad (13b)$$

where

$$M_n = \sum_{j=0}^{n} \frac{(-1)^j n!}{(n-j)!\, j!\, (j-\frac{1}{2})!} \frac{\eta^{2j}}{2^j}.$$

It can be shown that equation (13a) converges for $t > 0$. The convergence of (13b) can easily be shown by comparison with the geometric series.

The temperature distribution may also be evaluated using the integral expression of equation (11). Thus

$$T = \frac{B}{(4\pi k \rho c)^{\frac{1}{2}}} \int_0^t \frac{t'^i \exp\{-[\rho c y^2/4k(t - t')]\}}{(t - t')^{\frac{1}{2}}} dt', \quad t \leqslant p, \quad i > -1 \quad (14a)$$

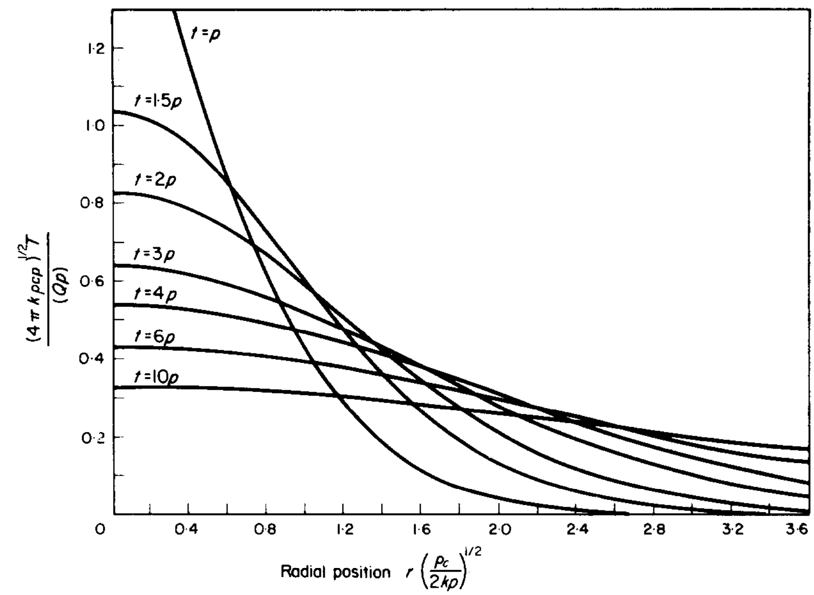

FIG. 1. Temperature distribution at various times after the end of constant heat generation.

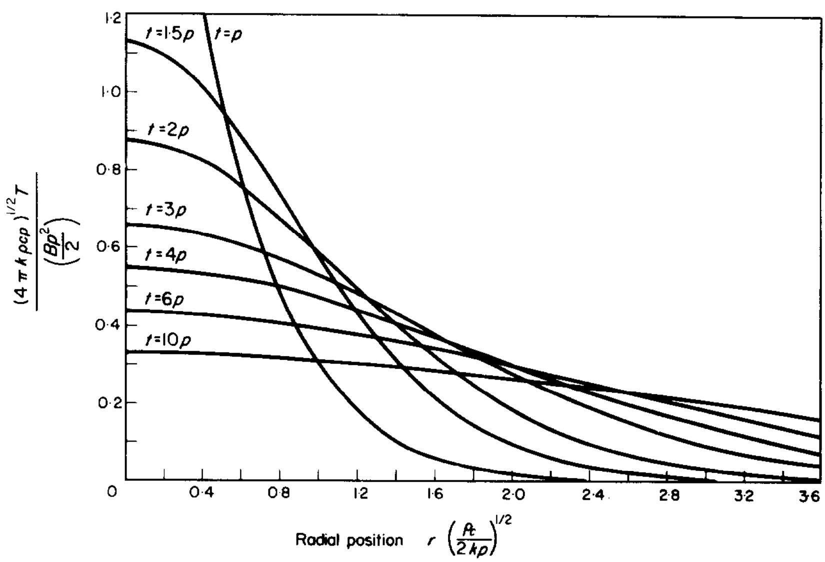

FIG. 2. Temperature distribution at various times after the end of ramp heat generation.

$$T = \frac{B}{(4\pi k\rho c)^{\frac{1}{2}}} \int_0^p \frac{t'^i \exp\{-[\rho c y^2/4k(t-t')]\}}{(t-t')^{\frac{1}{2}}} \mathrm{d}t', \quad t > p, \quad i > -1. \tag{14b}$$

For $p/t \geq 1$, the given solutions [equations (13a) and (14a)] are equivalent to the classical exponential integral solution for a continuous plane source. For $p/t < 1$, the solution is that of a continuous source which then decays when the source is shut off.

For the case where $i = 0$ (constant heat generation), the integrated expression of equation (14a) is given on page 263 of [1]. Equations (14a) and (14b) can also be integrated by a term by term integration of the series expansion of the integrand. The resulting expression for equation (14a), for $t \leq p$, is found to be easier to use than equation (13a) because the former converges faster. For the same reason, equation (13b) for $t > p$, is more convenient to use than equation (14b). Furthermore, the involved numerical integration for various values of p as given by equation (14b) is avoided by using the more accessible parametric representation of equation (13b). The results of the integration of the latter at various values of p/t for the important case $i = 0$, is shown in Fig. 1 and for the case of ramp heat generation ($i = 1$), is shown in Fig. 2. A comparison of the two graphs shows that for the same period of heat generation and for the same amount of heat generated ($Bp^2/2 = Qp$), the temperature near the origin is higher for the case of ramp heat generation. For larger values of $t(t > 10\,p)$, the graphs show no appreciable difference between the corresponding temperature profiles.

ACKNOWLEDGEMENT

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A FORCED CONVECTIVE HEAT TRANSFER INCLUDING DISSIPATION FUNCTION AND COMPRESSION WORK FOR A CLASS OF NONCIRCULAR DUCTS

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NOMENCLATURE

a, physical length, introduced in (9) and (16) with different meanings;
A, area of cross-section of duct;
b, physical length, introduced in (16);
c_p, specific heat referred to mass and constant pressure;
c_1, c_2, c_4, $(1/\mu)(\mathrm{d}p/\mathrm{d}z^*)$, $\rho c_p \tau/k$, $c_1 \cdot c_2$ respectively;
c_3, c_3^*, Q/k, c_3/c_4a^2 respectively;
D, domain of cross-section of duct;
De, hydraulic diameter, $4A/s$;
$E(\sqrt{1-\lambda^2})$, complete elliptic integral of second kind;
$f(z), g(z)$, functions of z, introduced in (3) and (5) respectively;
h, heat transfer coefficient;
k, thermal conductivity coefficient;
L, boundary of D;
Nu, Nusselt number;
p, pressure where $\mathrm{d}p/\mathrm{d}z^*$ is a negative constant;
q, heat transfer rate at the wall;
Q, intensity of heat source (or sink) distribution in the fluid medium;